Variational principle and free falling in a space-time with torsion

Rolando Gaitan D.[†], Juan Petit and Alfredo Mejía

[†]*Grupo de Física Teórica, Departamento de Física, Facultad de Ciencias y Tecnología, Universidad de Carabobo, A. P. 129 Valencia 2001, Edo. Carabobo, Venezuela.*

Abstract

A comparison between the two possible variational principles for the study of a free falling spinless particle in a space-time with torsion is noted. It is well known that the autoparallel trajectories can be obtained from a variational principle based on a non-holonomic mapping, starting with the standard world-line action. In a contrast, we explore a world-line action with a modified metric, thinking about the old idea of contorsion (torsion) potentials. A fixed-ends variational principle can reproduce autoparallel trajectories without restrictions on space-time torsion. As an illustration we have considered a perturbative Weitzenböck space-time. The non-perturbative problem is stablished at the end.

1 Introduction

From a variational principle point of view, there exist two ways for the study of trajectories of test particles in space-times with curvature and torsion. A non-holonomic mapping can be considered in the variations of a standard particle action which provides the expected autoparallel trajectory[1]. There, the fixed ends are not allowed and the failure of the closure of parallelograms depends directly on the non-null torsion. Moreover, a very particular shape of functional variations of coordinates (i.e.,"knock variations"), must be considered in order to obtain autoparallel trajectories.

On the other hand, for example, the proposal of an action constructed with the conformally transformed metric through the 0-spin part of torsion[2] is well known and, considering the conserved Belinfante tensor and the Papapetrou idea[3], it is possible to obtain the autoparallel equation of motion if a restriction on torsion is considered. This fact is noted when a standard fixed-ends variational principle is performed in the aforementioned type of action.

The main purpose of this work is to show that it would be possible to extend the idea about modified metric, beyond a conformal transformation, in the construction of a test particle action which could reproduce the autoparallel equation of motion without restrictions on space-time torsion. This modified metric shall consider a conformal transformation plus auxilliary fields (i.e., some type of torsion potentials). The introduction of torsion potentials is not new[4].

This paper is organized as follows. A brief review of the variational principle for the particle world-line action based on a non-holonomic mapping is performed in the next section. In section 3 we underline that there is no consistent autoparallel equation of motion arising from the test particle action constructed with a conformally modified metric and when a fixed-ends variational principle takes place. This means the appearance of serious restrictions on space-time torsion. However, we discuss the introduction of another modified metric which avoids any torsion restrictions and reproduce autoparallels considering a standard variational principle. In this sense, we tackle the most simple case as the Weitzenb$\ddot{o}$ck space-time with a weak torsion and, an interpretation from torsion potentials idea is performed. Suggestion on the non-perturbative formulation is explored thinking about some type of holonomies of a connection based on $GL(N,R) \times GL(N,R)$ for the expression of the modified metric. We end with some remarks.

2 A non-holonomic map

A brief review on a special variational principle for a spinless test particle in a space-time with torsion[1] starts here. Let ${e_\mu}^a$ be the vielbein which (locally) connects curvilinear, x^μ $(\mu = 0, \ldots, N-1)$ and Lorentzian, ξ^a $(a = 0, \ldots, N-1)$ coordinates, in other words

$$ {e_\mu}^a(x) = \left.\frac{\partial \xi^a}{\partial x^\mu}\right|_x \ , \tag{1} $$

and this allows us to (locally) relate the metric $g_{\mu\nu}$ with the Minkowski one, η_{ab} as follows

$$ g_{\mu\nu} = {e_\mu}^a {e_\nu}^b \eta_{ab} \ . \tag{2} $$

Indexes $\mu, \nu, \lambda, \ldots$ and $a, b, c, \ldots$ are raised and lowered with the corresponding metrics and $\eta = diag(-1, 1, 1, \ldots, 1)$. The non-holonomic behavior of the map $e : \xi^a \rightarrow x^\mu$ which is realized through the Jacobian elements (1), is established by means of the fact that the first does not satisfy the Schwarz condition (i.e., $\partial_\mu {e_\nu}^a - \partial_\nu {e_\mu}^a \neq 0$) in presence of torsion. So, from now on we asume an non-Riemannian space-time with null metricity, this means

$$ D_\mu {e_\nu}^a = \partial_\mu {e_\nu}^a - {(A_\mu)^\lambda}_\nu {e_\lambda}^a + {(\omega_\mu)^a}_b {e_\nu}^b = 0 \ , \tag{3} $$

where A_μ and ω_μ are the affine and spin connections, respectively. Using (3), one can write down the Schwarz condition as follows

$$ {e^\rho}_a [\partial_\mu {e_\nu}^a - \partial_\nu {e_\mu}^a] = {T^\rho}_{\mu\nu} - {t^\rho}_{\mu\nu} \neq 0 \ , \tag{4} $$

where ${T^\rho}_{\mu\nu} \equiv {(A_\mu)^\rho}_\nu - {(A_\nu)^\rho}_\mu$ and ${t^\rho}_{\mu\nu} \equiv {(\omega_\mu)^\rho}_\nu - {(\omega_\nu)^\rho}_\mu$ are the components of the torsion tensors (affine and spin, respectively). The affine connection is related with the Christoffel symbols through

$$ {(A_\mu)^\lambda}_\nu = {\Gamma^\lambda}_{\mu\nu}(g) + {K^\lambda}_{\mu\nu} \ , \tag{5} $$

defining the Christoffel symbols as always, ${\Gamma^\lambda}_{\mu\nu}(g) = \frac{g^{\rho\lambda}}{2}[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}]$ and ${K^\lambda}_{\mu\nu} = \frac{1}{2}({T^\lambda}_{\mu\nu} + {T_\mu}^\lambda{}_\nu + {T_\nu}^\lambda{}_\mu)$ are the contorsion tensor components.

Let τ be an invariant parameter which allows us to describe the particle trajectory. Then, one can write down the integral form of the inverse transformation espressed in (1), in the way

$$x^\mu(\tau) = x^\mu(\tau_0) + \int_{\tau_0}^{\tau} d\tau' e^\mu{}_a(\tau')\dot{\xi}^a(\tau') \ . \tag{6}$$

If arbitrary functional variations on coordinates are performed (i.e, $x'^\mu = x^\mu + \delta x^\mu$ and $\xi'^a = \xi^a + \delta\xi^a$), from (6) we can write $\delta x^\mu(\tau) = e^\mu{}_a(\tau)\delta\xi^a(\tau) - e^\mu{}_a(\tau_0)\delta\xi^a(\tau_0) + \int_{\tau_0}^{\tau} d\tau'\partial_\alpha e^\mu{}_a[\dot{\xi}^a\delta x^\alpha - \dot{x}^\alpha\delta\xi^a]$. Thinking in a fixed starting point (i.e., $\delta\xi^a(\tau_0) = 0$), we have

$$\delta x^\mu(\tau) = e^\mu{}_a(\tau)\delta\xi^a(\tau) + \int_{\tau_0}^{\tau} d\tau'[e_\alpha{}^a\partial_\nu e^\mu{}_a - e_\nu{}^a\partial_\alpha e^\mu{}_a]\dot{x}^\alpha\delta x^\nu \ . \tag{7}$$

Here, $e^\mu{}_a\delta\xi^a$ is the holonomic variation of x^μ, whereas δx^μ is the non-holonomic one. According to (7) it is confirmed the non-holonomic aspect of the map $e : \xi^a \to x^\mu$ because this fact depends on the contribution of the difference of affine and spin torsions, on virtue of (4). In the next discussion we shall assume a null spin torsion (i.e., $t^\rho{}_{\mu\nu} = 0$).

Now, we review the world-line action for a test particle

$$S = \frac{m}{2}\int_{\tau_0}^{\tau} d\tau' g_{\mu\nu}(x(\tau'))\dot{x}^\mu(\tau')\dot{x}^\nu(\tau') \ , \tag{8}$$

and considering the "knock" variation, $e^\mu{}_a(\tau)\delta\xi^a(\tau) = \epsilon^\mu(\tau)\delta(\tau - \tau_0)$, the variation of the action (8) is $\delta S = -m\epsilon_\mu\left[\ddot{x}^\mu + (A_\alpha)^\mu{}_\beta\dot{x}^\alpha\dot{x}^\beta\right]$. For arbitray functions $\epsilon_\mu(\tau)$, the extremal of the action conduce to the well known autoparallel equation

$$\ddot{x}^\mu + (A_\alpha)^\mu{}_\beta\dot{x}^\alpha\dot{x}^\beta = 0 \ . \tag{9}$$

Then, one can question if it is viable to construct a test particle action which can reproduce (9) under a fixed-ends variational principle. This is the main purpose of the next section.

3 A modified metric

It is well known the way to obtain autoparallel equations of motion through the Papapetrou metod[3]. However, there exist some conditions which constraint the background (i.e., 0-spin component of torsion is not arbitrary). This aspect can be stressed from the point of view of a fixed-ends variational principle of the action[2]

$$S = \frac{m}{2} \int_{\tau_0}^{\tau} d\tau' e^{-\phi(x(\tau'))} g_{\mu\nu}(x(\tau')) \dot{x}^{\mu}(\tau') \dot{x}^{\nu}(\tau') \ , \tag{10}$$

where $\phi(x)$ would be thought as an arbitrary scalar field. Notation is introduced for the conformally modified metric, $\overline{g}_{\mu\nu}(x) = e^{-\phi(x)} g_{\mu\nu}(x)$, then the extremal of (10) provide the following equation of motion

$$\ddot{x}^{\mu} + \Gamma^{\mu}{}_{\alpha\beta}(\overline{g}) \dot{x}^{\alpha} \dot{x}^{\beta} = 0 \ , \tag{11}$$

with

$$\Gamma^{\mu}{}_{\alpha\beta}(\overline{g}) = \Gamma^{\mu}{}_{\alpha\beta}(g) - \frac{1}{2}(\delta^{\mu}{}_{\alpha}\partial_{\beta}\phi + \delta^{\mu}{}_{\beta}\partial_{\alpha}\phi - g_{\alpha\beta}\partial^{\mu}\phi) \ . \tag{12}$$

If one claim that (11) describe autoparallel trajectories, then the following N constraints on contorsion (torsion) components must arise

$$N K_{\mu\alpha}{}^{\alpha} + (N-2) K^{\alpha}{}_{\{\mu\alpha\}} = 0 \ , \tag{13}$$

where N is the dimension of the space-time and $K^{\lambda}{}_{\{\alpha\beta\}} \equiv \frac{1}{2}(K^{\lambda}{}_{\alpha\beta} + K^{\lambda}{}_{\beta\alpha})$. As an illustration, let us consider a $2+1$ dimensional space-time where, if $K_{\mu\nu} = K_{\nu\mu}$ is a symmetric tensor and $\varepsilon^{\alpha\beta\lambda}$ is the Levi-Civita tensor; then the contorsion tensor can be decomposed as $K^{\lambda}{}_{\alpha\beta} = \varepsilon^{\sigma\lambda}{}_{\nu} K_{\mu\sigma} + \delta^{\lambda}{}_{\mu} V_{\nu} - g_{\mu\nu} V^{\lambda}$. Using the last expression and (12) in (13) conduce to $V_{\alpha} = 0$ and $\phi = 0$, which is a serious restriction on the model.

Next, we explore an extension of the modified metric definition and the action is given by

$$S = \frac{m}{2} \int_{\tau_0}^{\tau} d\tau' \widetilde{g}_{\mu\nu}(x(\tau')) \dot{x}^{\mu}(\tau') \dot{x}^{\nu}(\tau') \ , \tag{14}$$

where notation means

$$\widetilde{g}_{\mu\nu}(x) = e^{-\phi(x)} g_{\mu\nu}(x) + h_{\mu\nu}(x) + \mathcal{O}^{2}{}_{\mu\nu}(\phi, h)(x) \ . \tag{15}$$

Here, $\phi(x)$ and $h_{\mu\nu}(x)$ are scalar and symmetric rank-two field, respectively. The additional term $\mathcal{O}^2{}_{\mu\nu}(\phi, h)(x)$ represents all possible contributions on higher order of auxiliary fields. The extremized action (14) leads to

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}(\widetilde{g})\dot{x}^\alpha\dot{x}^\beta = 0 \ , \tag{16}$$

and if we claim that this equation describes autoparallels, the following condition appears

$$K^\mu{}_{\{\alpha\beta\}} = \Gamma^\mu{}_{\alpha\beta}(\widetilde{g}) - \Gamma^\mu{}_{\alpha\beta}(g) \ . \tag{17}$$

The main task is to solve (17) obtaining possible solutions for auxiliary fields, $\phi(x)$ and $h_{\mu\nu}(x)$ in terms of the semi-symmetric components of contorsion and without constraints on the last one.

As an illustration, let us consider a Weitzenb$\ddot{o}$ck space-time with a weak torsion and the natural relationship, $K^\lambda{}_{\mu\nu} \to 0 \Rightarrow \phi \to 0, h_{\mu\nu} \to 0$. At first order we write

$$\widetilde{g}_{\mu\nu} = \eta_{\mu\nu} - \phi\eta_{\mu\nu} + h_{\mu\nu} \ , \tag{18}$$

$$\widetilde{g}^{\mu\nu} = \eta^{\mu\nu} + \phi\eta^{\mu\nu} - h^{\mu\nu} \ , \tag{19}$$

where the indexes are lowered or raised with the Minkowski metric and $\widetilde{g}_{\mu\nu}\widetilde{g}^{\mu\lambda} = \delta_\nu{}^\lambda$. With the help of (18) and (19), we can rewrite (17) as follows

$$K^\mu{}_{\{\alpha\beta\}} = \gamma^\mu{}_{\alpha\beta}(h) - \frac{1}{2}(\delta^\mu{}_\alpha\partial_\beta\phi + \delta^\mu{}_\beta\partial_\alpha\phi - \eta_{\alpha\beta}\partial^\mu\phi) \ , \tag{20}$$

where $\gamma^\mu{}_{\alpha\beta}(h) \equiv \frac{\eta^{\mu\sigma}}{2}[\partial_\alpha h_{\sigma\beta} + \partial_\beta h_{\sigma\alpha} - \partial_\sigma h_{\alpha\beta}]$. A trace of (20) gives us the first derivative of scalar field ϕ, in other words

$$\partial_\beta\phi = \frac{2}{N}(K^\mu{}_{\{\mu\beta\}} - \gamma^\mu{}_{\mu\beta}(h)) \ , \tag{21}$$

and (20) is now

$$\begin{aligned} &\gamma_{\mu\alpha\beta}(h) - \frac{1}{N}\big(\eta_{\mu\beta}\gamma^\sigma{}_{\sigma\alpha}(h) + \eta_{\mu\alpha}\gamma^\sigma{}_{\sigma\beta}(h) - \eta_{\alpha\beta}\gamma^\sigma{}_{\sigma\mu}(h)\big) \\ &= K_{\mu\{\alpha\beta\}} - \frac{1}{N}\big(\eta_{\mu\beta}K^\sigma{}_{\{\sigma\alpha\}} + \eta_{\mu\alpha}K^\sigma{}_{\{\sigma\beta\}} - \eta_{\alpha\beta}K^\sigma{}_{\{\sigma\mu\}}\big) \ . \end{aligned} \tag{22}$$

It can be observed that the right-hand side of (22) is the traceless part of the semi-symmetric contorsion; this means $K^t{}_{\mu\{\alpha\beta\}} \equiv K_{\mu\{\alpha\beta\}} - \frac{1}{N}\big(\eta_{\mu\beta}K^\sigma{}_{\{\sigma\alpha\}} + \eta_{\mu\alpha}K^\sigma{}_{\{\sigma\beta\}} - \eta_{\alpha\beta}K^\sigma{}_{\{\sigma\mu\}}\big)$ with $K^{t\mu}{}_{\{\mu\beta\}} \equiv 0$. So, if we write the traceless part of the field $h_{\mu\nu}$ in the way $h_{\alpha\beta} = h^t{}_{\alpha\beta} + \frac{\eta_{\alpha\beta}}{N}\, h^\sigma{}_\sigma$ with $h^{t\sigma}{}_\sigma \equiv 0$, equation (22) is

$$\gamma_{\mu\alpha\beta}(h^t) = K^t{}_{\mu\{\alpha\beta\}} \ , \tag{23}$$

saying that the trace of $h^\sigma{}_\sigma$ remain unsolved. The solution for $h^t{}_{\alpha\beta}$ from (23) is a straightforward task, in other words

$$\partial_\mu h^t{}_{\alpha\beta} = K^t{}_{\alpha\{\beta\mu\}} + K^t{}_{\beta\{\alpha\mu\}} \ . \tag{24}$$

Up to a fixation of the trace of $h_{\mu\nu}$, it is possible to obtain all the auxiliary fields using (21) and (24). In a linearized regime, it would be sufficient to choose $\widetilde{g}_{\mu\nu} = \eta_{\mu\nu} - \phi\eta_{\mu\nu} + h^t{}_{\mu\nu}$ instead of (18).

In the non-perturbative case, equation (17) represents a homogeneous first order diferential equations system and, assuming a non singular metric $\widetilde{g}_{\mu\nu}$, one can write

$$\partial_\beta \widetilde{g}_{\lambda\alpha} - \big[\delta^\sigma{}_\lambda (A_{\{\alpha)})^\mu{}_{\beta\}} + \delta^\sigma{}_\alpha (A_{\{\lambda)})^\mu{}_{\beta\}}\big]\widetilde{g}_{\sigma\mu} = 0 \ . \tag{25}$$

This equation suggests solutions for $\widetilde{g}_{\mu\nu}$ via some type of holonomies of some connection and, then an algebraic problem for the solutions of auxiliary fields (potentials) ϕ and $h_{\mu\nu}$, considering (15). This business related to holonomies can be focused from the following point of view. If one considers composed indexes (i.e., Petrov indexes), A, B, C,...= $\mu\nu$ and the elements of the transformation group for greek ones are $U \in GL(N,R)$, then, if the transformation rule for the metric is $\widetilde{g}'_{\mu\nu} = U^\alpha{}_\mu U^\beta{}_\nu \widetilde{g}_{\alpha\beta}$, it can be written as follows

$$\widetilde{g}'_A = \mathcal{U}^B{}_A \widetilde{g}_B \ , \tag{26}$$

where $\mathcal{U}^B{}_A \equiv U^\alpha{}_\mu U^\beta{}_\nu \in \mathcal{G} \equiv GL(N,R) \times GL(N,R)$ and the metric $\widetilde{g}_B$ is arranged like a vector with N^2 components. Next, let $(\mathcal{A}_\beta)^{\sigma\mu}{}_{\lambda\alpha} \equiv (\mathcal{A}_\beta)^A{}_B$ be a field defined by

$$(\mathcal{A}_\beta)^{\sigma\mu}{}_{\lambda\alpha} \equiv \delta^{\{\sigma}{}_\lambda (A_{\{\alpha)})^{\mu\}}{}_{\beta\}} + \delta^{\{\sigma}{}_\alpha (A_{\{\lambda)})^{\mu\}}{}_{\beta\}} \ , \tag{27}$$

equation (25) is now

$$\partial_\mu \widetilde{g}_A - (\mathcal{A}_\mu)^B{}_A \widetilde{g}_B = 0 \ , \tag{28}$$

and covariance of this equation is satisfied if $\mathcal{A}_\mu$ transforms as a connection under $\mathcal{G}$, this means

$$(\mathcal{A}_\mu)^{B'}{}_{A'} = (\mathcal{U}^{-1})^B{}_C (\mathcal{A}_\mu)^C{}_D \mathcal{U}^D{}_A + (\mathcal{U}^{-1})^B{}_C \partial_\mu \mathcal{U}^C{}_A \ . \tag{29}$$

Let ${P_{x_o}}^x$ be an oriented path which starts at "x_o" and ends at "x", the solution of (28) is

$$\widetilde{\mathbf{g}}(x) = \mathrm{P}\, exp\big(\int_{{P_{x_o}}^x} dx^\beta \mathcal{A}_\beta \big) \ \widetilde{\mathbf{g}}(x_o) \ , \tag{30}$$

where $\widetilde{\mathbf{g}}$ is the N^2-vector with components $\widetilde{g}_A$ and the symbol "P" means path ordered integration. Then, solutions for $\widetilde{g}_{\alpha\beta}$ are coming from holonomies with algebraic constraints (i. e., $(\mathcal{A}_\beta)^{\sigma\mu}{}_{\lambda\alpha} = (\mathcal{A}_\beta)^{\mu\sigma}{}_{\lambda\alpha} = (\mathcal{A}_\beta)^{\sigma\mu}{}_{\alpha\lambda}$).

4 Conclusion

A possible action for a massive spinless particle when its free falling does take place in a space-time with torsion, has been introduced. Considering the old idea of contorsion (torsion) potentials, the interaction is provided through a modified metric and, by means of a standard variational principle the autoparallel equation of motion can be recovered without restrictions on contorsion (torsion). This fact is illustrated taking into account the simplest and not trivial case as it is the Weitzenböck space-time with a weak torsion (contorsion). There, the potential character of auxilliary fields ϕ and $h_{\mu\nu}$ is revealed when they are solved from direct integration of semi-symmetric contorsion components. At a non-perturbative regime, the situation is not different and the modified metric can be obtained from a certain class of holonomies of the connection $\mathcal{A}_\mu$, defined due to some algebraic constraints. The task related to particles with spin in a free falling in space-times with torsion must be considered elsewhere.

References

[1] H. Kleinert, P. Fiziev, *Europhys. Lett.* **35** (1996) 241.

[2] G. Aprea, G. Montani, R. Ruffini, *Int. Journ. Mod. Phys.* **D12** (2003) 1875.

[3] A. Papapetrou, *Proc. Roy. Soc. Lond.* **A209** (1951) 248.

[4] R. Hammond, *Gen. Rel. Grav.* **26** (1994) 247.